JOURNAL OF COMPUTATIONAL PHYSICS 65, 57–70 (1986)

Semi-implicit Method for Long Time Scale Magnetohydrodynamic Computations in Three Dimensions

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Received May 28, 1985; revised August 23, 1985

A semi-implicit method for solving the 3-dimensional magnetohydrodynamic equations on long time scales is presented. Standard explicit methods must use time steps which are constrained by a Courant–Friedrichs–Lewy condition due to the fast compressional and shear alfven motion. This semi-implicit method eliminates both of these restrictions so that very large time steps are permitted. The method is simple to implement and the computation time for one time step is essentially the same as for explicit methods. Numerical test results in slab and cylindrical geometry are presented. © 1986 Academic Press, Inc.

I. INTRODUCTION

The magnetohydrodynamic (MHD) equations are used extensively to study the macroscopic behavior of plasmas [1, 2]. Three-dimensional time-dependent computations are difficult due to the presence of widely disparate time scales in the MHD equations. Explicit methods are forced to use time steps limited by a very restrictive Courant–Friedrichs–Lewy (CFL) condition imposed by the fast compressional (magnetosonic) motion. Two-dimensional MHD computations using implicit methods have been performed by a number of authors (e.g., Schnack and Killeen [2], DeLucia and Jardin [3]). Implicit schemes for the 3-dimensional MHD equations have also been developed [4] but have generally had the disadvantages of being very complicated to implement and requiring the solution of

* Supported under U.S. Department of Energy Contract DE-AC02-76ER03077.
[†] Supported under U.S. Department of Energy Contract DE-AC03-83ER53150.

0021-9991/86 $3.00

large block matrix equations. A common approach to eliminate the fast compressional CFL condition is to make analytic approximations such as in the reduced equations [5–7] (inverse aspect ratio expansion) or incompressible models. Unfortunately, in many problems (e.g., incompressible reversed-field pinch dynamo computations [8]) important features of the physical system are also eliminated by these approximations. Recently, two different methods have been developed in order to solve the full compressible MHD equations in three dimensions, but without a CFL time step restriction due to the fast modes. One approach is that of Aydemir and Barnes [9] where an implicit pressure advance is used to eliminate the fast mode constraint. The other method is the semi-implicit method of Harned and Kerner [10, 11] in which new terms are added to the time discretized MHD equations. These new terms do not affect the solution as $\Delta t \to 0$, but still produce a method that is unconditionally stable with respect to the fast modes.

Once the fast compressional time step constraint has been eliminated, the time step is then limited by a shear Alfven CFL constraint. This constraint is particularly severe in the case of the reversed-field pinch, in which the toroidal and poloidal magnetic fields are comparable. In tokamak plasmas the shear Alfven CFL condition becomes very restrictive in the nonlinear phase of resistive instabilities. The methods of [9–11], as well as incompressible and reduced equation methods, all require a substantial reduction in time step in the highly nonlinear stages of resistive instabilities, which ultimately limit their capabilities.

Semi-implicit methods are well suited for use in eliminating the overly restrictive constraints of ideal MHD, because they are very flexible and simple to implement. In a semi-implicit method, terms which approximate the linear behavior of fast normal modes are treated implicitly in order to enhance the numerical stability of the method. Although these methods may be used effectively with grid point models, they are most powerful when used in conjunction with a spectral representation. In large scale meteorological calculations semi-implicit methods were first introduced [12] to eliminate the severe time step constraint due to external gravity waves. These methods are used extensively now in meteorology [13–16] to produce accurate computations with much larger times steps than are permitted in explicit methods, yet without the added complexity of implicit algorithms. In this paper we develop a semi-implicit algorithm for the long time scale numerical solution of the 3-dimensional MHD equations by generalizing the method of Harned and Kerner [10, 11] to remove the shear Alfven time step restriction.

The method to be described here is designed to eliminate the shear Alfven time step restriction as well as that due to the fast compressional waves. It is much simpler to implement than an implicit method and the computation time required for a time step is essentially the same as that for an explicit method. Section II describes the model and the semi-implicit method. The results of numerical tests are presented in Section III and conclusions are given in Section IV.

II. Semi-implicit MHD Method

The compressible resistive MHD equations may be written

$$\frac{\partial \bar{v}}{\partial t} = -(\bar{v} \cdot \nabla)\bar{v} + \frac{1}{\rho}[(\nabla \times \bar{B}) \times \bar{B} - \nabla P] \tag{1}$$

$$\frac{\partial \bar{A}}{\partial t} = \bar{v} \times \bar{B} - \eta \nabla \times \nabla \times \bar{A} \tag{2}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \bar{v}) \tag{3}$$

$$\frac{\partial P}{\partial t} = -\bar{v} \cdot \nabla P - \gamma P \nabla \cdot \bar{v} + \text{dissipative terms} \tag{4}$$

$$\bar{B} = \nabla \times \bar{A} \tag{5}$$

where $\bar{v}$ is the velocity, $\bar{B}$ the magnetic field, P the plasma pressure, ρ the density, and η the resistivity. $\bar{A}$ is the vector potential and the gauge condition $\phi = 0$ has been used. Resistive instabilities are important in the analysis of fusion plasmas. However, they evolve on a time scale which is long compared to ideal time scales (e.g., fast compressional and shear Alfven). Therefore, in order to solve Eqs. (1)–(5) numerically, it is desirable to develop a method that is not forced to use time steps constrained by the rapid ideal motion.

To remove the time step constraint imposed by the fast compressional modes a semi-implicit method was used in [10, 11]. Because of the flexibility of this method it is natural to try to extend it to eliminate the shear Alfven time step constraint as well. In a semi-implicit method one advances all of the terms in the original equations explicitly. New terms are added to the time discretized equations which do not affect the solution in the limit $\Delta t \to 0$ (i.e., the method is still consistent). Then some of these "semi-implicit" terms are treated implicitly. As an example consider the simple hyperbolic system,

$$\frac{\partial u}{\partial t} = a \frac{\partial v}{\partial x} \tag{6}$$

$$\frac{\partial v}{\partial t} = a \frac{\partial u}{\partial x}. \tag{7}$$

These equations may be rewritten

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}. \tag{8}$$

To develop a semi-implicit method, a new term is subtracted from each side of Eq. (8),

$$\frac{\partial^2 u}{\partial t^2} - a_0^2 \frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial x^2} - a_0^2 \frac{\partial^2 u}{\partial x^2}; \tag{9}$$

a_0 is a constant coefficient and will be chosen primarily from numerical stability considerations. Equation (9) is then time differenced as

$$u^{n+1} - a_0^2 (\Delta t)^2 \frac{\partial^2 u^{n+1}}{\partial x^2} = u^n + a\Delta t \left(\frac{\partial u}{\partial t}\right)^n + a^2 (\Delta t)^2 \frac{\partial^2 u^n}{\partial x^2} - a_0^2 (\Delta t)^2 \frac{\partial^2 u^n}{\partial x^2}. \tag{10}$$

A practical way to time advance Eqs. (6) and (7) in order to obtain an algorithm equivalent to Eq. (10) is with a two step predictor-corrector

$$u^* = u^n + \alpha a \, \Delta t \frac{\partial v^n}{\partial x} \tag{11}$$

$$v^* = v^n + \alpha a \, \Delta t \frac{\partial u^n}{\partial x} \tag{12}$$

$$u^{n+1} - a_0^2 (\Delta t)^2 \frac{\partial^2 u^{n+1}}{\partial x^2} = u^2 + a \, \Delta t \frac{\partial v^*}{\partial x} - a_0^2 (\Delta t)^2 \frac{\partial^2 u^n}{\partial x^2} \tag{13}$$

$$v^{n+1} = v^n + (a/2) \, \Delta t \left(\frac{\partial u^n}{\partial x} + \frac{\partial u^{n+1}}{\partial x} \right) \tag{14}$$

with $0.5 \leqslant \alpha \leqslant 1.0$. This method is unconditionally stable if $a_0^2 > (a^2/16)(1 + 2\alpha)^2$. For a simple 1-dimensional case like this the semi-implicit method offers no advantages over an implicit method. However, if a problem is 3-dimensional and spectral in two dimensions with nonconstant coefficients, an implicit method is difficult because it requires performing convolutions in the implicit terms. This leads to a complicated matrix equation. In the semi-implicit method the constant coefficient semi-implicit terms require no convolutions; therefore, only a simple tridiagonal solution is needed. Another important feature of the method is the use of the second-order equations for discretization. If the method were applied to the original first-order equations, the sign of a becomes important. If a changes sign, the constant coefficient semi-implicit term would not stabilize the system because in the region where a and a_0 are of opposite sign the semi-implicit term would actually be destabilizing. However, in the second-order equation, Eq. (9), the coefficient only enters as a^2 so that the sign does not matter and the semi-implicit treatment becomes very simple.

In [10, 11] the fast compressional time step constraint was eliminated by choosing a semi-implicit term having the same form as the fast modes. This term was subtracted from each side of a second-order equation. Then the MHD equations, with the new semi-implicit terms included, were differenced in a predictor-corrector scheme analogous to Eqs. (11)–(14). To extend the method to also eliminate the shear Alfven time scale, we must derive a new semi-implicit term. First, the MHD equations are linearized, assuming a uniform magnetic field, density, and pressure. Then, after some algebra, Eqs. (1)–(5) reduce to

$$\frac{\partial^2 \bar{v}}{\partial t^2} = [\nabla \times \nabla \times (\bar{v} \times \bar{B}_0)] \times \bar{B}_0 + \gamma P_0 \nabla(\nabla \cdot \bar{v}). \tag{15}$$

To arrive at a semi-implicit term, $\bar{B}_0$ is replaced with a vector quantity with constant coefficients, $\bar{C}_0 = C_x \hat{x} = C_y \hat{y} + C_z \hat{z}$, where C_x, C_y, and C_z are all constant in space. The pressure term is dropped, because it enters in the magnetosonic modes with the same form as the perpendicular magnetic field. Therefore, the semi-implicit term is just

$$[\nabla \times \nabla \times (\bar{v} \times \bar{C}_0)] \times \bar{C}_0.$$

One now naively hopes that this term could be used in a predictor-corrector method to eliminate the shear Alfven time step constraint in the same way as the fast mode time step constraint was eliminated. This philosophy suggests the following algorithm. First, a simple explicit predictor advance is performed,

$$\bar{v}^* = \bar{v}^n + \alpha\, \Delta t \frac{1}{\rho^n} \bar{F}(\rho, \bar{v}, \bar{B}, P)^n \tag{16}$$

$$\bar{A}^* = \bar{A}^n + \alpha\, \Delta t \bar{v}^n \times \bar{B}^n \tag{17}$$

$$P^* = P^n - \alpha\, \Delta t (\bar{v}^n \cdot \nabla P^n + \gamma P^n \nabla \cdot \bar{v}^n) \tag{18}$$

$$\rho^* = \rho^n - \alpha\, \Delta t \nabla \cdot (\rho \bar{v})^n \tag{19}$$

$$\bar{B}^* = \nabla \times \bar{A}^* \tag{20}$$

where $\bar{F}$ represents the right side of Eq. (1). The semi-implicit term is included in the corrector velocity advance:

$$v^{n+1} - \frac{(\Delta t)^2}{\rho^*} [\nabla \times \nabla \times (\bar{v}^{n+1} \times \bar{C}_0)] \times \bar{C}_0$$
$$= \bar{v}^n + \frac{\Delta t}{\rho^*} \bar{F}(\rho, \bar{v}, \bar{B}, P)^* - \frac{(\Delta t)^2}{\rho^*} [\nabla \times \nabla \times (\bar{v}^n \times \bar{C}_0)] \times \bar{C}_0. \tag{21}$$

The new velocity is used in the magnetic field, pressure, and density advances,

$$\bar{A}^{**} = \bar{A}^n + \frac{\Delta t}{2} [(\bar{v}^{n+1} + \bar{v}^n) \times \bar{B}^*] \tag{22}$$

$$P^{n+1} = P^n - \frac{\Delta t}{2} [(\bar{v}^{n+1} + \bar{v}^n) \cdot \nabla P^* + \gamma P \nabla \cdot (\bar{v}^{n+1} + \bar{v}^n)] \tag{23}$$

$$\rho^{n+1} = \rho^n - \frac{\Delta t}{2} \nabla \cdot [\rho^* (\bar{v}^{n+1} + \bar{v}^n)]. \tag{24}$$

Finally, a time step is completed with a semi-implicit resistive advance

$$A^{n+1} + \Delta t \eta_0 \nabla \times \nabla \times A^{n+1} = A^{**} - \Delta t \eta \nabla \times \nabla \times A^{**} + \Delta t \eta_0 \nabla \times \nabla \times A^{**}. \tag{25}$$

The resistive part of the MHD equations has been split from the ideal part and treated semi-implicitly so that the resistive advance does not impose an additional timestep constraint. As in the Harned–Kerner algorithm, [11] the treatment is semi-implicit rather than implicit so that nonuniform resistivity may be used without requiring convolutions to be performed in the implicit part.

It is hoped that if a set of conditions like $C_x \gtrsim B_x$, $C_y \gtrsim B_y$, and $C_z \gtrsim B_z$ is satisfied, the method would be unconditionally stable with respect to both the fast compressional and shear Alfven modes. For a 1-dimensional case a linear stability analysis shows that this is true. However, the critical difference between this method and that used for the fast modes is that $\bar{C}_0$ is a vector rather than a scalar. Unfortunately, one finds for a 2-dimensional stability analysis that unconditional stability for the fast and shear modes can occur only when $\bar{C}_0$ is parallel to $\bar{B}$. This means that when $\bar{B}$ is nonuniform, $\bar{C}_0$ must be nonuniform as well. This makes the method equivalent to an implicit method, requiring the solution of complicated matrix equations for the time advance. Such a method is impractical and defeats the purpose of the semi-implicit method.

To solve this problem, it is important to realize that the form of the semi-implicit terms is completely arbitrary (although extreme choices may seriously degrade the accuracy of the method). The terms should be chosen to enhance the numerical stability of the method, yet still be simple to treat implicitly. In a 2-dimensional stability analysis, the terms having semi-implicit coefficients $C_i C_j$, with $i \neq j$, are destabilizing when $\bar{C}_0$ and $\bar{B}_0$ are not parallel. When these terms are set to zero, an unconditionally stable algorithm results. In addition, this new modified algorithm has the desirable property that the semi-implicit coefficients enter only as C_i^2, so that the sign of the coefficients does not matter.

To demonstrate the linear stability properties of the method, consider a 2-dimensional cold ideal case with a uniform density and equilibrium magnetic field. The MHD equations reduce to

$$\frac{\partial \bar{v}}{\partial t} = (\nabla \times \bar{B}) \times \bar{B} \tag{26}$$

$$\frac{\partial \bar{A}}{\partial t} = \bar{v} \times \bar{B}. \tag{27}$$

Let $\bar{A} = A_z \hat{z}$, $\bar{v} = v_x \hat{x} + v_y \hat{y}$, and $\partial/\partial z = 0.0$. Then a method similar to that of Eqs. (16)–(24), with $\alpha = 1.0$, is

$$\bar{v}^* = \bar{v}^n + \Delta t(\nabla \times \bar{B}^n) \times \bar{B}^n \tag{28}$$

$$A_z^* = A_z^n + \Delta t(\bar{v}^n \times \bar{B}^n)_z \tag{29}$$

$$\begin{aligned} \bar{V}^{n+1} &- (\Delta t)^2[\nabla \times \nabla \times (\bar{v}^{n+1} \times \bar{C}_0)] \times \bar{C}_0 \\ &= \bar{V}^n + \Delta t(\nabla \times B^*) \times B^* - (\Delta t)^2[\nabla \times \nabla \times (\bar{v}^n \times \bar{C}_0)] \times \bar{C}_0 \end{aligned} \tag{30}$$

$$A_z^{n+1} = A_z^n + \Delta t(v^{n+1} \times B^*)_z. \tag{31}$$

except that Eq. (30) is modified by setting the terms with $C_x C_y$ equal to zero. The spatial differencing is computed with a spectral representation,

$$f(x) = \sum_{m,n} f_{mn} e^{imx + iny}.$$

The system, Eqs. (28)–(31) with the $C_x C_y$ terms dropped, may be written as the matrix equation

$$\begin{bmatrix} 1 + N^2 C_y^2 & 0 & 0 \\ 0 & 1 + N^2 C_x^2 & 0 \\ \Delta t B_y & \Delta t B_x & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ A_z \end{bmatrix}^{n+1} = \begin{bmatrix} 1 + N^2(C_y^2 - B_y^2) & B_x B_y N^2 & -B_y N^2/\Delta t \\ B_x B_y N^2 & 1 + N^2(C_x^2 - B_x^2) & B_x N^2/\Delta t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ A_z \end{bmatrix}^{n} \tag{32}$$

where N^2 is defined as $N^2 = (m^2 + n^2)(\Delta t)^2$. Equation (32) is then rewritten in the form

$$\begin{bmatrix} v_x \\ v_y \\ A_z \end{bmatrix}^{n+1} = G \begin{bmatrix} v_x \\ v_y \\ A_z \end{bmatrix}^{n} \tag{33}$$

where G is the amplification matrix,

$$G=\begin{bmatrix} 1-Y & \frac{B_x}{B_y}Y & -Y/B_y \\ \frac{B_y}{B_x}X & 1-X & X/B_x \\ B_y(1-X-Y) & -B_x(1-X-Y) & 1-X-Y \end{bmatrix}. \tag{34}$$

In Eq. (34) X and Y are defined by

$$X \equiv N^2 B_x^2/(1+N^2C_x^2)$$

and

$$Y \equiv N^2 B_y^2/(1+N^2C_y^2).$$

To determine the linear stability of the algorithm, the eigenvalues of the amplification matrix are computed. In addition to the trivial result, $\omega=1$, they are

$$\omega = 1 - Z \pm \sqrt{Z(Z-1)} \tag{35}$$

where $Z=X+Y$. When $Z<\frac{4}{3}$ the eigenvalues lie inside the unit circle. Therefore, as desired, the method is unconditionally stable as long as $C_x \gtrsim B_x$ and $C_y \gtrsim B_y$. This is not the case if the C_xC_y cross terms are retained in the semi-implicit corrector. Although $\alpha \geqslant 0.5$ is always required, when α is reduced below $\alpha=1.0$ in the predictor, the stability condition, $Z<\frac{4}{3}$, is relaxed and somewhat smaller values of the semi-implicit coefficients may be used.

The complete semi-implicit predictor-corrector algorithm that eliminates both the shear Alfven and fast compressional CFL time step constraints is given by Eqs. (16)–(25), with the crucial modification that all terms with C_iC_j are replaced by $C_iC_j\delta_{ij}$ in Eq. (21). Solving this system is not difficult. A spectral representation is used in two directions. In the x direction (or the radial direction for cylindrical coordinates) centered finite differences are used. All of the quantities in the ideal advance are advanced explicitly with the exception of the velocity corrector. In the velocity corrector the semi-implicit term on the left-hand side is treated implicitly, but no convolutions are required. Therefore, this equation is a block tridiagonal matrix equation with 3×3 blocks. This equation may be solved easily without adding any significant amount of computation time per time step, when compared to an explicit method. This is because for any case with at least a few modes, the computation time is dominated by the convolutions. As the plasma evolves nonlinearly in time the coefficients, C_i, may be varied in time (and in space, if desired) to preserve linear stability with respect to the new values of magnetic field and pressure. The method described here is first-order accurate in time for $\alpha>0.5$. The algorithm of Eqs. (16)–(24) may be made second-order accurate in time by adding one additional corrector step in which the semi-implicit terms are all at the new time level. A similar procedure is used by Joyce [17] in a second-order semi-implicit algorithm for beam propagation problems. Although second-order accuracy in time is not critical for explicit methods due to their small time steps, in semi-implicit methods which may use very large time steps it can provide a significant improvement over a first-order method.

III. Numerical Tests

The new semi-implicit algorithm described in Section II has been tested in slab and cylindrical geometry to demonstrate its numerical stability properties. In a slab we assume a constant density, $\rho = 1.0$. The y and z directions are periodic and are treated with a spectral representation. Finite differences are used in the x direction. Rigid conducting well boundary conditions are imposed at $x = 0.0$ and $x = 1.0$. First, a uniform magnetic field is initialized with $B_z = 1.0$ and $B_y = 0.2$. We apply the following 3-dimensional perturbation with $m = 1$ and $n = 1$, where δ is the perturbation amplitude:

$$v_x = (0.96/2\pi)\delta \sin(my + 0.2nz) \sin(2\pi x)$$

$$v_y = 1.0\delta \cos(my + 0.2nz) \cos(2\pi x)$$

$$v_z = -0.2\delta \cos(my + 0.2nz) \cos(2\pi x).$$

This is an excitation of a shear Alfven wave with frequency $\omega = \bar{k} \cdot \bar{B} = 0.4$. We use 41 grid points and keep modes from $m = 1$, $n = 1$ to $m = 20$, $n = 20$. Although the wave is linear, the code is nonlinear so that all of the modes are excited. If we do not use the semi-implicit method, i.e., set $\bar{C}_0 = 0.0$, then numerical instability results of the time step exceeds the usual CFL condition on the fast modes, $\Delta t < \Delta x / v_a = 0.025$. Furthermore, even if the fast mode time step constraint is eliminated, the shear Alfven time step constraint limits the time step to $\Delta t < 0.24$. To simulate the Alfven wave we first choose a time step of $\Delta t = 0.1$ which is four [illegible]

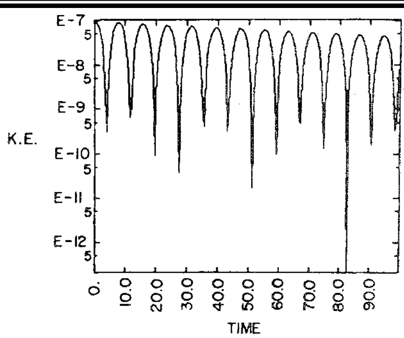

FIG. 1. Kinetic energy of a shear Alfven wave due to a 3-dimensional perturbation, with $\omega = 0.4$ and $\Delta t = 0.1$. The semi-implicit method properly simulates the wave even though the time step is four times the usual explicit CFL limit.

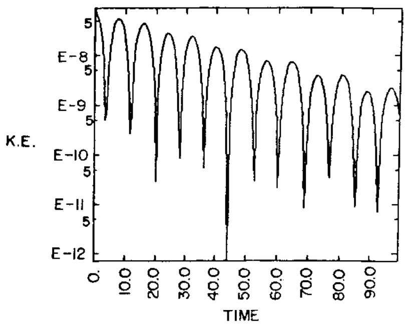

FIG. 2. Kinetic energy of the wave of Fig. 1 except with $\Delta t = 0.5$. This time step is more than twice the usual shear Alfven CFL limit for this 20-mode case. The frequency is correct although damping is present due to the predictor-corrector scheme.

method is set to $\alpha = 0.6$. We use $C_z = 0.7$ and $C_y = 0.2$. The kinetic energy of the wave is shown in Fig. 1. The expected period of $\tau = 5\pi$ is produced correctly. In Fig. 2 we show the result of a simulation with $\Delta t = 0.5$, more than twice the shear Alfven limit. Some damping of the wave kinetic energy is apparent due to $\alpha = 0.6$ in the predictor-corrector method. The wave frequency is still correct. As the time step is increased further, the method remains stable but the wave becomes poorly represented. In fact, the time step can be raised to extremely large values, such as $\Delta t = 200$, and the algorithm will be stable but the wave is immediately damped. We note that if the cross terms (e.g., $C_x C_y$) in the semi-implicit terms are retained, the method fails, as expected.

As a second test we use a uniform magnetic field, $B_z = 1.0$, and apply a 2-dimensional perturbation with

$$v_y = \delta \cos(ny) \cos(2\pi x).$$

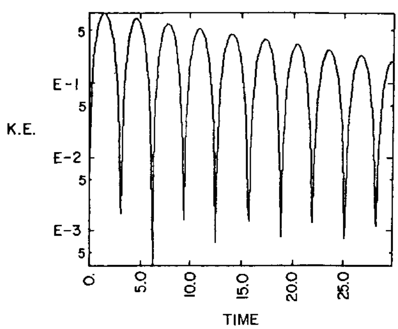

FIG. 3. Kinetic energy for a shear Alfven wave produced by a large amplitude perturbation. For this many mode case the time step, $\Delta t = 0.1$, violates the usual stability conditions not only for the equilibrium field, B_z, but also for the large amplitude perturbed field, B_y. Nevertheless, using the semi-implicit method stability is ensured and the correct frequency, $\omega = 1.0$, is obtained.

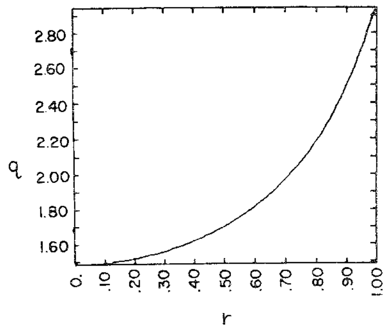

FIG. 4. Safety factor, q, as a function of radius for a cylindrical tokamak equilibrium which is unstable to an $m=2$, $n=1$ resistive instability.

Twenty modes with $m=0$ are retained with the largest being $n=200$. An additional small random perturbation is also applied so that all of the modes are excited. Setting $C_z=0.7$ and $C_y=0.0$ would be sufficient to have an unconditionally stable algorithm for a linear wave. However, if the wave has a large amplitude, then B_y becomes substantial and it may impose a stability limit as well. For a case when the perturbed $B_y=0.3$ and $\Delta t=0.1$, the method goes immediately numerically unstable. However, when $C_y=0.3$ is used unconditional stability again results. The kinetic energy for the case with $C_z=0.7$, $C_y=0.3$, and $\Delta t=0.1$ is shown in Fig. 3. The algorithm is stable even with the large amplitude oscillatory magnetic field, B_y, present and the correct frequency is obtained.

In cylindrical geometry we initialize an equilibrium with $B_z=1.0$, $J_z=J_0(1-r^2/a^2)$, and $k_z=0.33$. The safety factor at the wall, $q(a)=2\pi r B_z(a)/LB_\theta(a)$, is set to $q(a)=2.93$. The safety factor for this profile is shown in Fig. 4. The resistivity is set to $\eta=10^{-5}$; 200 radial grid points are used and twenty modes from $m=2$, $n=1$ to $m=40$, $n=20$ are retained. The time step is set to $\Delta t=1.5$ radial Alfven transit times. The normal CFL limit for an explicit code would be $\Delta t<0.005$ for the

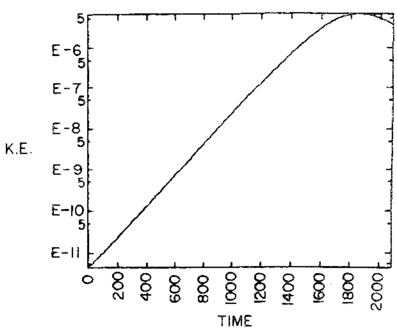

FIG. 5. Kinetic energy as a function of time for the unstable equilibrium of Fig. 4. The time step exceeds the usual CFL limit for the shear Alfven modes by more than a factor of ten, yet the growth rate is accurate to within a few percent.

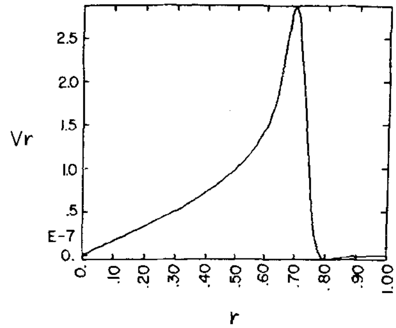

FIG. 6. Radial velocity profile for the $m=2$, $n=1$ mode during linear growth from the equilibrium of Fig. 4.

fast modes and $\Delta t < 0.12$ for the shear Alfven modes. Hence, both of these conditions have been violated by more than a factor of ten. For this case we allow the semi-implicit coefficients to vary with radius and time. At each time step the components of $\bar{C}_0$ are set to 0.9 times the maximum value of their respective magnetic field components at a given radius. A random perturbation is applied so that the fastest growing unstable mode should eventually dominate. We expect to observe an $m=2$, $n=1$ instability with a growth rate of $\gamma=0.0046$. The kinetic energy as a function of time is plotted in Fig. 5. The growth rate obtained from the simulation is $\gamma=0.0044$, giving good agreement. The profiles of v_r and J_z are shown in Figs. 6 and 7, respectively. In Fig. 8 are plotted helical flux contours, showing the $m=2$, $n=1$ island near the saturation of linear growth. The time step may be increased further. However, very large time steps degrade the accuracy of the results. A time step of five Alfven times, $\Delta t=5.0$, using the first-order accurate method gives a substantially smaller growth rate for the instability, $\gamma=0.0028$, and the velocity and current profiles show a departure from the proper eigenfunctions. When we use the second-order accurate algorithm we obtain an accurate value for the growth rate,

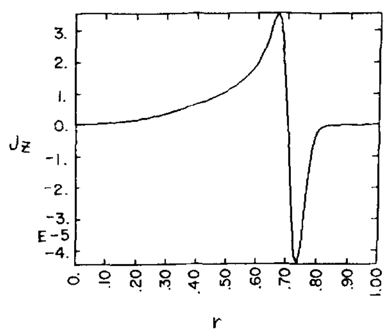

FIG. 7. J_z profile for the $m=2$, $n=1$ mode during linear growth from the equilibrium of Fig. 4.

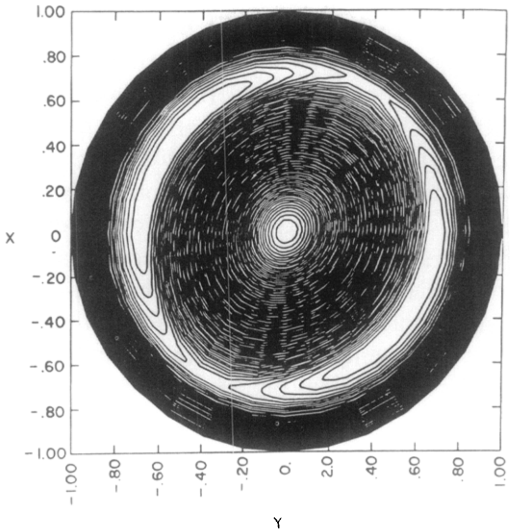

FIG. 8. Helical flux contours for the case of Fig. 4 showing the $m=2$, $n=1$ island near the saturation of linear growth.

$\gamma = 0.0044$, even with $\Delta t = 5.0$. If still larger time steps are used, such as $\Delta t = 10.0$, the method remains stable, but the accuracy for even the second-order accurate method is poor. Typically we find that accuracy begins to deteriorate for tearing mode computations when the time step becomes comparable to the Alfven transit time down the length of the cylinder. In some highly nonlinear problems, such as tokamak disruptions in which the flow velocity may approach the Alfven velocity, advection can impose an important stability restriction. In such cases, without an implicit treatment of the advective terms, the time step must be reduced accordingly. In terms of computing time, we note that for the preceding tearing mode calculation with 20 modes, the computation time required for a time step using the first-order accurate method is only 5% more than for an explicit advance, even though the time step may be more than 300 times larger. If the second-order accurate method is used the computation time required per time step is 50% more than for an explicit method.

IV. Conclusions

A new semi-implicit method for solving the MHD equations in three dimensions has been developed. The method allows simulations to use very large time steps since both the shear Alfven and the fast compressional CFL conditions have been eliminated. The method is simple to implement. The most difficult part of the time advance is the solution of a block tridiagonal system of 3×3 blocks. Therefore, the method requires virtually the same amount of computation time per time step as an explicit method.

The method has been tested on simple problems in slab and cylindrical geometry, which verify the unconditional stability of the algorithm with respect to both shear Alfven and fast compressional modes. The primary limiting factor on a time step now appears to be accuracy. In the algorithm of [11], the results using first-order and second-order accurate methods are essentially identical. In that algorithm time steps were limited by the shear Alfven CFL condition and were therefore sufficiently small so that error due to the time discretization would be negligible. However, in the present algorithm the stability limits permit very large time steps so that the time discretization error becomes significant. For this reason it is advantageous to implement the algorithm with second-order accuracy in time. It is now desirable to apply the new method to modeling the nonlinear evolution of resistive instabilities in realistic tokamak and reversed-field pinch equilibria.

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